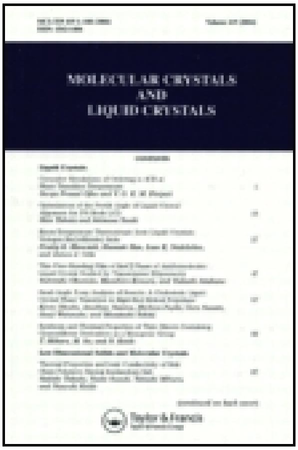

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We present a theoretical investigation of the director orientation induced by an electric field in a wedge of smectic C liquid crystal. A complex coupling exists between the liquid crystal and the field and we have incorporated this feature in this problem. The coupling behaviour between the electric field and the liquid crystal has a noticeable impact on the orientation. Numerical results for the director orientation and the influenced electric field are presented.

Keywords: field effects; Freedericksz transitions; smectic *C*

INTRODUCTION

The influence of an electric or magnetic field on a liquid crystalline material is well documented [1,2]. When applying such a field to a planar sample of nematic liquid crystal, the liquid crystal director will start to rotate, due to competing boundary and bulk orientations, providing a certain threshold field (Freedericksz threshold) is reached. A similar effect occurs in samples of smectic *C* (SmC) liquid crystals. The main difference arises from the fixed layered structure of SmC in which the director is forced to rotate around a ficticious cone defined by the smectic tilt angle θ [2]. Rapini [3] has investigated various transitions in planar geometries and later work by Leslie *et al.* [4,5] led to further research into planar, wedge, cylindrical and spherical geometries. Aspects of these cases are discussed by Carlsson *et al.* [6], Kedney and Stewart [7] and Atkin and Stewart [8–10].

Usually, when dealing with applied electric fields, it is assumed that the fluid does not interact with the field, i.e. it is assumed uniform. For magnetic fields there is negligible interaction. However, this is not in

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general the case for an electric field. Deuling [11] looked at this problem in a planar nematic cell and found a significant effect on the distortion configuration. Here this idea will be extended to the effect in a wedge of SmC liquid crystal.

THE WEDGE GEOMETRY

The wedge geometry is known to be a resourceful research vehicle for determining properties of SmC liquid crystals [6,8,12]. It consists of SmC sandwiched between two angled plates, across which a voltage, U, is applied so as to obtain an azimuthal field. In this configuration (see Fig. 1) β denotes the wedge angle, $\mathbf{n}$ the usual director, $\mathbf{n_b}$ the director at the boundaries, $\mathbf{a}$ the unit layer normal, $\mathbf{E}$ the electric field, and (r, α, z) define the cylindrical polar coordinate system. The c-director, $\mathbf{c}$, is introduced to represent a unit vector which is parallel to the projection of $\mathbf{n}$ onto the smectic planes and is described by the angle ϕ. For convenience (see Fig. 2), a unit vector $\mathbf{b}$ which lies within the smectic layers is introduced via $\mathbf{b} = \mathbf{a} \times \mathbf{c}$ [6].

From the SmC continuum theory [4,5], it is known that the c-director and the layer normal $\mathbf{a}$ are subject to the following constraints, assuming that there are no defects present

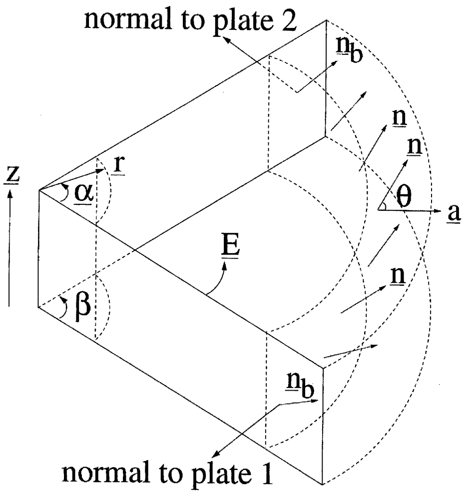

FIGURE 1 Director orientation in a wedge of SmC.

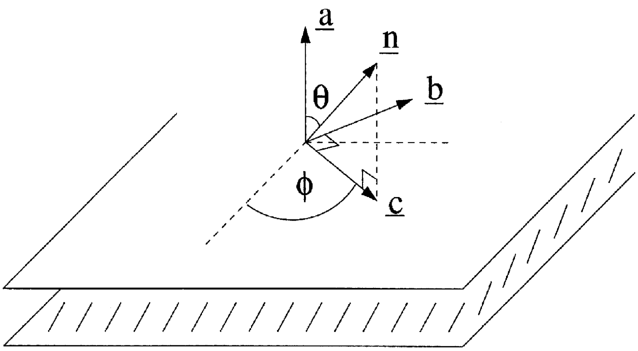

FIGURE 2 Local description of the smectic tilt angle θ, director **n** and the vectors **a**, **b** and **c**.

$$\mathbf{a}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{c}=1, \quad \mathbf{a}\cdot\mathbf{c}=0, \quad \nabla\times\mathbf{a}=\mathbf{0}. \tag{1}$$

Due to the symmetry of the problem there should be no dependence on the z coordinate and, following the work of Carlsson *et al.* [6], the r-dependence is neglected, resulting in $\phi=\phi(\alpha)$. It is also assumed that the unperturbed configuration when no field is present corresponds to $\phi(\alpha)\equiv 0$ [6] so that the director is given by

$$\mathbf{n}=\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\mathbf{z}}. \tag{2}$$

Strong anchoring is imposed on the boundary plates,

$$\phi(0)=\phi(\beta)=0. \tag{3}$$

By use of the above assumptions it is now possible to write down the following ansatz for **a**, **b** and **c** [6]

$$\mathbf{a}=\hat{\mathbf{r}}, \quad \mathbf{b}=-\cos\phi\hat{\boldsymbol{\alpha}}+\sin\phi\hat{\mathbf{z}}, \quad \mathbf{c}=\sin\phi\hat{\boldsymbol{\alpha}}+\cos\phi\hat{\mathbf{z}}, \tag{4}$$

with $\phi=\phi(\alpha)$. The director **n** is then given by [6]

$$\mathbf{n}=\cos\theta\hat{\mathbf{r}}+\sin\theta\sin\phi\hat{\boldsymbol{\alpha}}+\sin\theta\cos\phi\hat{\mathbf{z}}. \tag{5}$$

FREEDERICKSZ TRANSITION IN THE WEDGE

The electric free energy density resulting from an electric field applied across a liquid crystal is approximated by [2, p. 134], [11]

$$w_{elec.}=-\frac{1}{2}\epsilon_o\epsilon_\perp E^2-\frac{1}{2}\epsilon_a\epsilon_o(\mathbf{n}\cdot\mathbf{E})^2=-\frac{1}{2}\mathbf{D}\cdot\mathbf{E}, \tag{6}$$

where $E = |\mathbf{E}|$, ϵ_o is the permittivity of free space, $\mathbf{D}$ is the electric displacement vector and $\epsilon_a = \epsilon_{\parallel} - \epsilon_{\perp}$ is the dielectric anisotropy, which is assumed to be positive, where $\epsilon_{\parallel}$ and $\epsilon_{\perp}$ are the components of permittivity along the director and perpendicular to the director respectively. A positive dielectric constant indicates that $\mathbf{n}$ is attracted to be parallel to the field. In this case, since the electric field is influenced by the director orientation, it is reasonable to assume

$$\mathbf{E} = \frac{F(\alpha)}{r}\hat{\boldsymbol{\alpha}}, \tag{7}$$

where $F(\alpha)$ is some function related to the voltage by $U = \int_0^{\beta} F(\alpha)d\alpha$. A good approximation to the electric displacement is

$$\mathbf{D} = \epsilon_o\epsilon_{\perp}\mathbf{E} + \epsilon_o\epsilon_a(\mathbf{n}\cdot\mathbf{E})\mathbf{n}. \tag{8}$$

Inserting Eqs. (5) and (7) into (8) gives

$$\mathbf{D} = \frac{\epsilon_o\epsilon_{\perp}F(\alpha)}{r}\hat{\boldsymbol{\alpha}} + \frac{\epsilon_o\epsilon_a F(\alpha)}{r}\sin\theta\sin\phi(\cos\theta\hat{\mathbf{r}} + \sin\theta\sin\phi\hat{\boldsymbol{\alpha}} + \sin\theta\cos\phi\hat{\mathbf{z}}). \tag{9}$$

Using the elastic energy derived by Carlsson *et al.* [6], and by use of (6), (7) and (9), the total energy over a sample of unit depth in z can be shown to be given by

$$\begin{aligned} W = \int_{\alpha}\int_{r}\Bigg[&-\frac{A_{11}}{2r} + \frac{1}{2r}\left[(A_{12}+A_{11})\sin^4\phi + (A_{21}+A_{11})\cos^4\phi\right] \\ &+\frac{1}{2r}\left[B_1\sin^2\phi + B_2\cos^2\phi\right]\left(\frac{d\phi}{d\alpha}\right)^2 \\ &+\frac{1}{r}\left[C_1\sin^2\phi - C_2\cos^2\phi\right]\cos\phi\frac{d\phi}{d\alpha} - \frac{CF(\alpha)}{2r}\Bigg]dr\,d\alpha, \end{aligned} \tag{10}$$

where the $A_{ij},\ B_i,\ B_{ij}$ and C_i represent the usual smectic elastic constants defined by the Orsay Group except $A_{11} = -\frac{1}{2}A_{11}^{Orsay}$ and $C_1 = -C_1^{Orsay}$ [5,6]. C is a constant of integration introduced via Maxwell's equations in their reduced form when no free charge is present ($\nabla\cdot\mathbf{D} = 0,\ \ \nabla\times\mathbf{E} = \mathbf{0}$) and is found to be given by the functional

$$C = \epsilon_o U\left\{\int_0^{\beta}\left[\epsilon_{\perp} + \eta\sin^2\phi\right]^{-1}d\alpha\right\}^{-1}, \tag{11}$$

which arises from $\nabla \cdot \mathbf{D} = 0$. Here η is the constant parameter, $\eta = \epsilon_a \sin^2 \theta$. By integrating from $r = r_o$ to $r = r_1$, the total energy per unit depth in the z direction is then

$$W = \frac{1}{2}\ln\left|\frac{r_1}{r_o}\right| \int_0^\beta \Big[-A_{11} + \left[(A_{12}+A_{11})\sin^4\phi + (A_{21}+A_{11})\cos^4\phi\right] + \left[B_1 \sin^2\phi + B_2 \cos^2\phi\right]\left(\frac{d\phi}{d\alpha}\right)^2 + 2\left[C_1 \sin^2\phi - C_2 \cos^2\phi\right]\cos\phi\left(\frac{d\phi}{d\alpha}\right)\Big] d\alpha - \frac{1}{2}\ln\left|\frac{r_1}{r_o}\right| \epsilon_o U^2 \left\{ \int_0^\beta \left[\epsilon_\perp + \eta \sin^2\phi\right]^{-1} d\alpha \right\}^{-1}. \tag{12}$$

Since this energy models a physical system, it is expected that for a given applied voltage the director configuration which occurs is that which requires least energy. The equilibrium equation for the above energy must be obtained by setting the usual first variation of W to zero. By multiplying the consequent differential equation throughout by $d\phi/d\alpha$, integrating, and imposing the conditions [8]

$$\left.\frac{d\phi}{d\alpha}\right|_{\alpha=\frac{\beta}{2}} = 0, \quad \phi\left(\frac{\beta}{2}\right) = \phi_m, \tag{13}$$

we obtain ϕ implicitly as a function of α

$$\alpha = \int_0^\phi \left[B_1 \sin^2\tilde{\phi} + B_2 \cos^2\tilde{\phi}\right]^{\frac{1}{2}} \left\{ \frac{C^2}{\epsilon_o}\left[\frac{\eta(\sin^2\phi_m - \sin^2\tilde{\phi})}{(\epsilon_\perp + \eta\sin^2\tilde{\phi})(\epsilon_\perp + \eta \sin^2\phi_m)}\right] + \left[(A_{12}+A_{11})(\sin^4\tilde{\phi} - \sin^4\phi_m) + (A_{21}+A_{11})(\cos^4\tilde{\phi} - \cos^4\phi_m)\right] \right\}^{-\frac{1}{2}} d\tilde{\phi}. \tag{14}$$

Following Atkin and Stewart [8, Eq. (34)], we must ensure that the integrand in (14) is non-negative: this is indeed the case for the data taken from Findon [12] and used in (22). Inserting the second condition from (13) then gives the following expression involving the wedge angle β:

$$\frac{\beta}{2} = \int_0^{\phi m} \left[B_1 \sin^2\phi + B_2 \cos^2\phi\right]^{\frac{1}{2}} \left\{ \frac{C^2}{\epsilon_o} \left[\frac{\eta(\sin^2\phi_m - \sin^2\phi)}{(\epsilon_\perp + \eta\sin^2\phi)(\epsilon_\perp + \eta\sin^2\phi_m)}\right] + \left[(A_{12} + A_{11})(\sin^4\phi - \sin^4\phi_m) + (A_{21} + A_{11})(\cos^4\phi - \cos^4\phi_m)\right] \right\}^{-\frac{1}{2}} d\phi. \tag{15}$$

It is now found convenient to introduce the substitution

$$\sin\phi = \sin\phi_m \sin\psi, \quad \text{provided } \psi, \phi_m \neq \frac{\pi}{2} \tag{16}$$

and the parameters $\mu = \sin^2\phi_m$, $\eta = (\epsilon_\| - \epsilon_\perp)\sin^2\theta$, $\rho = (B_1 - B_2)/B_2$ and $\gamma = (\epsilon_\| - \epsilon_\perp)/\epsilon_\perp$, so that Eqs. (14) and (15) become

$$\alpha = B_2^{\frac{1}{2}} \int_0^{\sin^{-1}\left(\frac{\sin\phi}{\sin\phi_m}\right)} \left[\frac{\rho\mu\sin^2\psi + 1}{1 - \mu\sin^2\psi}\right]^{\frac{1}{2}} \times \left\{ \frac{C^2}{\epsilon_o\epsilon_\perp} \left[\frac{\gamma\sin^2\theta}{(1 + \gamma\mu\sin^2\theta\sin^2\psi)(1 + \gamma\mu\sin^2\theta)}\right] - \left[(A_{12} + A_{11})\mu(\sin^2\psi + 1) + (A_{21} + A_{11})(\mu\sin^2\psi + \mu - 2)\right] \right\}^{-\frac{1}{2}} d\psi, \tag{17}$$

$$\frac{\beta}{2} = B_2^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{\rho\mu\sin^2\psi + 1}{1 - \mu\sin^2\psi}\right]^{\frac{1}{2}} \times \left\{ \frac{C^2}{\epsilon_o\epsilon_\perp} \left[\frac{\gamma\sin^2\theta}{(1 + \gamma\mu\sin^2\theta\sin^2\psi)(1 + \gamma\mu\sin^2\theta)}\right] - \left[(A_{12} + A_{11})\mu(\sin^2\psi + 1) + (A_{21} + A_{11})(\mu\sin^2\psi + \mu - 2)\right] \right\}^{-\frac{1}{2}} d\psi, \tag{18}$$

Ultimately, we use Eqs. (17) and (18) to find the unknowns C and ϕ_m and also the solution ϕ: we return to this below.

From the voltage obtained in Eq. (11), we can use (14) to change the variable of integration from α to ϕ and, assuming that the problem is symmetric about $\alpha = \beta/2$, it can be shown that

$$U = \frac{2C}{\epsilon_o} \int_0^{\phi_m} \frac{[B_1 \sin^2\phi + B_2 \cos^2\phi]^{\frac{1}{2}}}{[\epsilon_\perp + \eta \sin^2\phi]} \times \left\{ \frac{C^2}{\epsilon_o} \left[\frac{\eta(\sin^2\phi_m - \sin^2\phi)}{(\epsilon_\perp + \eta \sin^2\phi)(\epsilon_\perp + \eta \sin^2\phi_m)} \right] + \left[(A_{12} + A_{11})(\sin^4\phi - \sin^4\phi_m) + (A_{21} + A_{11})(\cos^4\phi - \cos^4\phi_m) \right] \right\}^{-\frac{1}{2}} d\phi. \tag{19}$$

Introducing (16) along with μ, η, ρ and γ, (19) becomes

$$U = \frac{C\beta}{\epsilon_o\epsilon_\perp} - \frac{2CB_2^{\frac{1}{2}}}{\epsilon_o\epsilon_\perp} \gamma\mu \sin^2\theta \int_0^{\frac{\pi}{2}} \frac{\sin^2\psi[\rho\mu\sin^2\psi + 1]^{\frac{1}{2}}}{[1 + \gamma\mu\sin^2\theta\sin^2\psi][1 - \mu\sin^2\psi]^{\frac{1}{2}}} \times \left\{ \frac{C^2}{\epsilon_o\epsilon_\perp} \left[\frac{\gamma \sin^2\theta}{(1 + \gamma\mu\sin^2\theta\sin^2\psi)(1 + \gamma\mu\sin^2\theta)} \right] - \left[(A_{12} + A_{11})\mu(\sin^2\psi + 1) + (A_{21} + A_{11})(\mu\sin^2\psi + \mu - 2) \right] \right\}^{-\frac{1}{2}} d\psi, \tag{20}$$

where Eq. (18) has been used to obtain the first term. Until a critical voltage U_c is reached, the distorted solution, $\phi(\alpha)$, is unobtainable and no director reorientation occurs. At $U = U_c$, however, the point is reached where $\phi_m = 0$ and is on the verge of changing. In order to calculate this Freedericksz threshold it is required to take the limit as $\mu \to 0$ (i.e. $\phi_m \to 0$) in Eqs. (18) and (20). By carrying out these limiting procedures it is found that U_c satisfies

$$\epsilon_a\epsilon_o U_c^2 \sin^2\theta = \pi^2 B_2 - 2\beta^2(A_{21} + A_{11}), \tag{21}$$

which is identical to that obtained earlier by Atkin and Stewart [8]. The changes caused by the interaction between the fluid and field should only come into effect after the Freedericksz threshold has been reached and the director begins to reorientate to align with the field.

POST-FREEDERICKSZ THRESHOLD SOLUTIONS

It is possible to solve numerically Eqs. (18) and (20) simultaneously in order to calculate the unknowns C and ϕ_m for a given set of parameter values as will be seen below. Once obtained, they can be inserted into Eq. (17) which then leads to an implicit solution $\phi(\alpha)$ for the director orientation.

To apply realistic data for the material parameters, attention has focussed on the experimental work of Findon [12] who made various measurements of the SmC material M3 (see [12] for the chemical structure of this particular phase). The following values are adopted:

$$A_{12} = 1.711 \times 10^{-4}\ \text{N},\ A_{11} = -1.44 \times 10^{-4}\ \text{N},\ A_{21} = 1.212 \times 10^{-4}\ \text{N},$$
$$\beta = 20 \times 10^{-4}\ \text{rads},\ B_1 = 7.02 \times 10^{-12}\ \text{N},\ B_2 = 3.51 \times 10^{-12}\ \text{N},$$
$$\epsilon_\perp = 2.91,\ \epsilon_\parallel = 3.95,\ \theta = \frac{2\pi}{15},\ \frac{U}{U_c} = 1.1, \tag{22}$$

which are taken from [12], except for A_{11}, A_{12}, A_{21} and B_1, which are consistent with the measurements in [12] and the inequalities in [6] and [8], and also $\frac{U}{U_c}$. It should be noted that Findon [12] measured the temperature independent contributions to the elastic constants which have been converted using the small θ approximations [6]

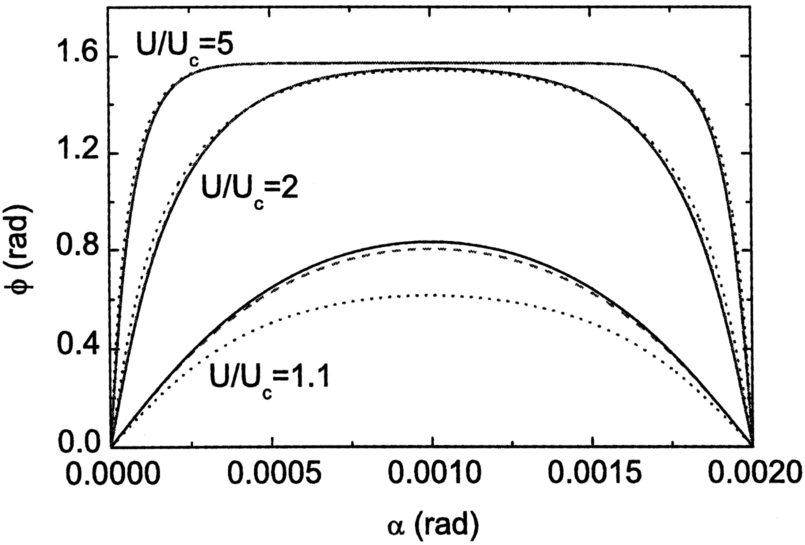

FIGURE 3 Distortion across a SmC wedge with uniform field (solid) and influenced field with $\epsilon_a = 1.04$ (dashed) and $\epsilon_a = 12$ (dotted) for various applied voltages. At higher voltage ratios all three curves overlap indicating that there is negligible difference when $U \gg U_c$.

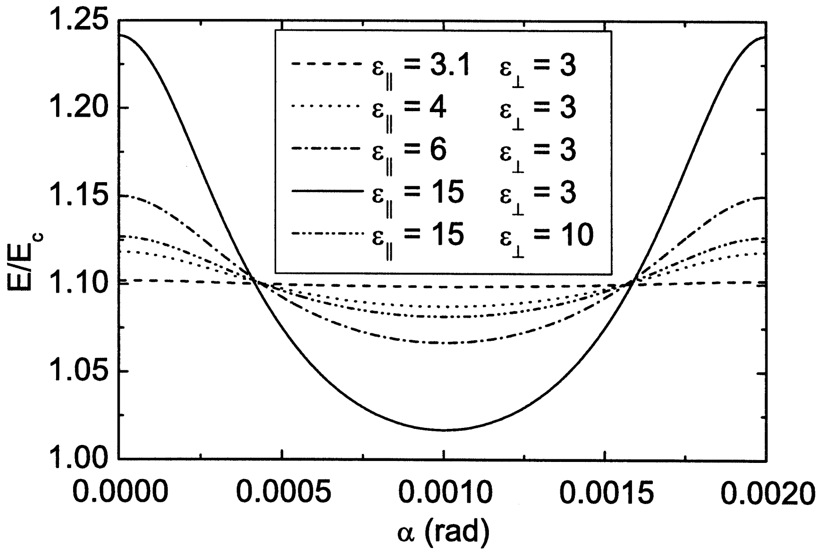

FIGURE 4 Electric field variation across a SmC wedge for differing permittivities.

$$(A_{12}+A_{11}) = (\bar{A}_{12}+\bar{A}_{11})\theta^2, \quad (A_{21}+A_{11}) = (\bar{A}_{21}+\bar{A}_{11})\theta^2,$$
$$B_1 = \bar{B}_1\theta^2, \qquad B_2 = \bar{B}_2\theta^2. \tag{23}$$

It is desirable to make a comparison with the work of Atkin and Stewart [8, Eqs. (36), (40)] who assumed a uniform electric field across the wedge when $U > U_c$.

Figure 3 shows a comparison of the solutions at different applied voltages for the uniform field assumption (solid lines) and the influenced field. The dashed lines represent the results for the data in (22); the dotted lines represent the solutions for $\epsilon_{\|} = 15$ and $\epsilon_{\perp} = 3$, giving $\epsilon_a = 12$, the other data remaining the same.

Making use of Eq. (7) along with the reduced Maxwell's equation, $\nabla \cdot \mathbf{D} = 0$, and the uniform field stated by Carlsson *et al.* [6] it can be shown that the ratio of the electric field to the Freedericksz threshold field is

$$\frac{E}{E_c} = \frac{C\beta}{\epsilon_o U_c\left(\epsilon_{\perp} + \eta \sin^2\phi\right)}, \tag{24}$$

where U_c is the threshold voltage defined by (21). This then allows the relative field strength to be plotted in order to see how it varies across the wedge. Figure 4 shows this variation for different permittivity values guided by the measurements in Dunmur *et al.* [13] where the data (22) has been used for the remaining parameters.

CONCLUSIONS

By extending the work of Deuling [11] on planar samples of nematic liquid crystals to the problem of a wedge of SmC liquid crystal it has been possible to determine the effect of an influenced electric field on the orientation of the director. For small values of the dielectric anisotropy and high voltage ratios there is negligible effect and the uniform field used by Atkin and Stewart [8] is a good approximation. For larger dielectric anisotropy values and voltages close to the threshold the effect on the orientation is more significant. The effect of varying the other material parameters can also be investigated and an energy analysis has been carried out to confirm that these solutions for the parameters stated in (22) are energetically favourable over the undistorted solution for $U > U_c$.

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